MATH/STAT 355: Problem Set 1

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Due: February 14, 2024

Probability Review

1. Suppose X and Y are random variables with probability mass functions $f_X(x) = \Pr(X = x)$ and $f_Y(y) = \Pr(Y = y)$, respectively. If X and Y are independent, write down an expression for the joint probability mass function of X and Y, $f_{X,Y}(x, y) = \Pr(X = x, Y = y)$.

2. Does your expression for $f_{X,Y}(x, y)$ in Question 1 hold if X and Y are not independent? Explain your answer.

3. Suppose that X and Y now represent independent flips of a coin (0 = tails, 1 = heads) with probability p of landing heads.

   (a) We can write: $f_X(x) = p^x(1-p)^{1-x}$ for $x = 0, 1$ and $f_Y(y) = p^y(1-p)^{1-y}$ for $y = 0, 1$. Explain why this makes sense in a couple of sentences.

   (b) The distributions of X and Y have a special name. What is it?

   (c) Write down an expression for $f_{X,Y}(x, y)$.

   (d) If $p = 0.7$, find the probability of observing $(X = 0, Y = 1)$.

   (e) If $p = 0.7$, what is the most likely (X, Y) sequence?

   (f) If $p = 0.4$, find the probability of observing $(X = 0, Y = 1)$.

   (g) If $p = 0.4$, what is the most likely (X, Y) sequence?

   (h) If you observed $(X = 0, Y = 1)$ and did not know the value of p, would you be more willing to believe that it is 0.7 or 0.4? Why?

4. Prove the following theorem: If the distribution of X is *symmetric* about 0, then the distribution of $Y = X^2$ can be written as $f_Y(y) = \frac{1}{\sqrt{y}} f_X(\sqrt{y})$. (hint: look at the proof in the course notes for transformations of variables to get started)

5. Use the theorem proved in Question 4 to show find the distribution of $Y = X^2$, where $X \sim N(0, 1)$. Take a look through the course notes and additionally determine which common distribution this is (with appropriate parameter values). You may wish to use the fact that $\Gamma(1/2) = \sqrt{\pi}$ when simplifying.

6. Suppose we have $X \sim Gamma(\alpha, \lambda)$ and $Y \sim Gamma(\beta, \lambda)$, where $X \perp\!\!\!\perp Y$. Let $U = X + Y$ and $W = \frac{X}{X+Y}$. Show that U and W are independent. (hint: find the joint pdf of (U, W), find the marginal pdfs of U and W, and proceed from there)

Maximum Likelihood Estimation

1. Suppose $X_1, X_2, \ldots, X_n$ are a random sample (i.e., the X_i's are independent and identically distributed) from the Exponential pdf with parameter β:

$$f_X(x;\beta) = \frac{1}{\beta} e^{-\frac{x}{\beta}},$$

for $x \geq 0$ and $\beta > 0$. Find the maximum likelihood estimator (MLE) of β.

2. Suppose $Y_1, \ldots, Y_n \overset{iid}{\sim} \text{Uniform}(0, \theta)$, so

$$f_Y(y;\theta) = \frac{1}{\theta}, \quad 0 \leq y \leq \theta.$$

Find the MLE of θ.

3. Suppose a random sample of size n is independently drawn from the probability model with pmf

$$p_X(x;\theta) = \frac{\theta^{2x} e^{-\theta^2}}{x!}, \quad x = 0, 1, 2, \ldots.$$

Find a formula for the maximum likelihood estimator, $\hat{\theta}$.

4. Find the maximum likelihood estimate for θ based on independently drawn data $Y_1, \ldots, Y_n$ from the pdf

$$f_Y(y) = \frac{2y}{1-\theta^2}, \quad 0 < \theta \leq y \leq 1$$

if a random sample of size 6 yielded the measurements 0.70, 0.63, 0.92, 0.86, 0.43, and 0.21.

5. Suppose $X_1, X_2, \ldots, X_n$ are a random sample from the Normal pdf with parameters μ and σ^2:

$$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2},$$

for $-\infty < x < \infty$, $-\infty < \mu < \infty$, and $\sigma^2 > 0$. Find the MLEs of μ and σ^2.

6. Suppose that $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, where β_0, β_1 are unknown; $x_1, \ldots, x_n$ are known; and the error terms $\epsilon_1, \ldots, \epsilon_n$ are a random sample from a Normal pdf with mean $\mu = 0$ and known variance $\sigma^2 > 0$. It follows that $Y_1, Y_2, \ldots, Y_n$ are independent with the pdf

$$f_Y(y_i;\beta_0,\beta_1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i-\beta_0-\beta_1 x_i)^2}.$$

Find the MLEs of β_0 and β_1.