MATH/STAT 355: Problem Set 3

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Properties of Estimators

1. Let $X_1, \dots, X_n \overset{iid}{\sim} Exponential(\lambda)$. Find a sufficient statistic for λ.

2. In this problem, we'll prove that
$$\left(\sqrt{\frac{n-1}{2}}\right)\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}S$$
is the UMVUE for σ, where $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$ and $X_i \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$.

   (a) Show that $\sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=1}^n (X_i - \mu)^2 - n(\bar{X} - \mu)^2$.

   (b) Show that $(\overline{X}, S^2)$ is a jointly sufficient statistic for (μ, σ^2).

   (c) Show that $\frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 - \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2$

   (d) Show that $\sum_{i=1}^n \left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_n$, and $\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2_1$. Use the known relationships that if $X \sim N(0,1)$, then $X^2 \sim \chi^2_1$, and that if, for $Y \perp\!\!\!\perp W$, $Y \sim \chi^2_n$ and $W \sim \chi^2_m$, $Y + W \sim \chi^2_{n+m}$.

   (e) Using everything we've done in the previous steps, show (finally) that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$.

   (f) Let $Y = \frac{(n-1)S^2}{\sigma^2}$, and note that $\sqrt{\frac{\sigma^2 Y}{n-1}} = S$. You may take without proof that $(\overline{X}, S^2)$ is a *complete*, jointly sufficient statistic for (μ, σ^2). Use RBLS to show that $\left(\sqrt{\frac{n-1}{2}}\right)\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}S$ is the UMVUE for σ.

Consistency

1. Prove Chebyshev's Corollary's

   (a) Corollary 1: If $\hat{\theta}_n$ is an unbiased estimator for θ and $\lim_{n\to\infty} Var(\hat{\theta}_n) = 0$, then $\hat{\theta}_n$ is consistent for θ.

   (b) Markov's Inequality, similar to Chebyshev's Inequality, states that if X is a non-negative random variable (i.e., $X \geq 0$), then $P(X \geq a) \leq \frac{E(X)}{a}$ for all $a > 0$. Use this inequality, along with the squeeze/sandwich theorem and the relationship between mean-squared error $E\left[(\hat{\theta}_n - \theta)^2\right]$, variance, and bias to prove Chebyshev's Corollary 2: If $\hat{\theta}_n$ is an *asymptotically* unbiased estimator for θ and $\lim_{n\to\infty} Var(\hat{\theta}_n) = 0$, then $\hat{\theta}_n$ is consistent for θ.

2. Let $Y_1, \dots, Y_n$ be a random sample of size n from a Normal pdf where $E(Y_i) = 0$ and variance $\text{Var}(Y_i) = \sigma^2$. Show that $S_n^2 = \frac{1}{n}\sum_{i=1}^n Y_i^2$ is a consistent estimator for σ^2.

Correlated Data: Part 1

Consider the following autoregressive model

$$X_n = \beta X_{n-1} + e_n, \quad n = 1, 2, 3, \ldots$$

with $-1 < \beta < 1$, $X_0 = 0$, and $e_1, \ldots, e_n$ i.i.d. with $E[e_i] = 0$ and $Var[e_i] = 1$ (Note that we have not assumed a pdf for e_i, so in this problem, this process is actually *semiparametric*). Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\overline{e}_n = \frac{1}{n}\sum_{i=1}^{n} e_i$.

If you've taken Correlated Data, you've seen this before! It's like an AR1! Wow, connections!

1. Calculate $Var(X_n)$.
2. Use Chebyshev's Inequality to show that $X_n/\sqrt{n} \xrightarrow{p} 0$. (Hint: $\sum_{i=0}^{n-1} \beta^{2i}$ is a convergent geometric series).

We'll do more with this problem on the next Problem Set!